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Variational Formulation of Approximate Symmetries and Conservation Laws

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We show how the conserved vectors and associated (approximate) Lie symmetry generators of a partial differential equation with a small parameter can be utilized to construct approximate Lagrangians for the equation. We then use the Lagrangian to further determine approximate Noether symmetries and, hence, new associated conservation laws. The theory is applied to a number of perturbations of the wave equation.

1. INTRODUCTION

It has been shown by Feroze and Kara (in press) that the procedure to construct Lagrangians for differential equations, using approximate symmetries and associated conservation laws along with Noether's theorem (see Noether, 1918), can be extended to ordinary differential equations with a small parameter (sometimes referred to as perturbed equations). Consequently, the symmetry and conserved quantity (first integral) used is a Noether approximate symmetry and first integral in the sense that the symmetry leaves invariant the functional that arises in the variational problem and the associated first integral satisfies Noether's theorem.

In the sequel, we extend the procedure to perturbed partial differential equations (p.d.e.). That is, approximate Lie symmetry generators and conserved vectors of a p.d.e. are used to determine Lagrangians for the equation (if these exist). As the Lagrangian is an approximate one, we maintain the order of the small parameter as it appears in the p.d.e.— in the examples here, the equations are first order in the small parameter.

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We now review some pertinent results. In Baikov *et al*. (1996), it is shown that if X_0 is a generator of Lie–Bäcklund symmetry of a partial differential equation

$$E_0^{\beta} = 0 \quad \beta = 1, \ldots, \tilde{m}, \tag{1.1}$$

then an *approximate Lie-Bäcklund symmetry*, $X = X_0 + \epsilon X_1$, of the perturbed partial differential equation

$$E_0^{\beta} + \epsilon E_1^{\beta} = 0 \tag{1.2}$$

is obtained by solving for X_1 in

$$X_1\big(E_0^{\beta}\big)\big|_{E_0^{\beta}=0} + H = 0, \tag{1.3}$$

where

$$H = \frac{1}{\epsilon} X_0\big(E_0^{\beta} + \epsilon E_1^{\beta}\big)\big|_{E_0^{\beta}+\epsilon E_1^{\beta}=0} \tag{1.4}$$

E_1^{β} is the perturbation and H is referred to as an auxilliary function. Further, an approximate conserved vector $T = (T^1, T^2)$ of (1.2) satisfies

$$D_i T^i|_{(1.2)} \;=\; O(\epsilon^2) \tag{1.5}$$

where

$$T^i = T_0^i + \epsilon T_1^i, \quad i = 1, 2. \tag{1.6}$$

Equation (1.5) is an approximate conservation law for (1.2). Furthermore, the components $T^i = T_0^i + \epsilon T_1^i$ of the approximate conserved vector, T, satisfies

$$\begin{aligned} X_0 T_0^i + D_j\big(\xi_0^j\big)T_0^i - T_0^j D_j\big(\xi_0^i\big) &= 0,\\ X_0\big(T_1^i\big) + D_j\big(\xi_0^j\big)T_1^i - T_1^j D_j\big(\xi_0^i\big) &= -\big(X_1\big(T_0^i\big) + D_j\big(\xi_1^j\big)T_0^i - T_0^j D_j\big(\xi_1^i\big)\big),\\ i &= 1, 2. \end{aligned} \tag{1.7}$$

For the Lagrangian formulation, more specifically for the inverse problem, we appeal to Noether's theorem. However, we first need to state the following theorem regarding the invariance of the functional in the variational problem. The proof is straightforward and proceeds in a way similar to the well known unperturbed case.

Theorem 1. *Suppose* $L(t, x, u, u_t, u_x, \epsilon) = L_0(t, x, u, u_t, u_x) + \epsilon L_1(t, x, u, u_t, u_x)$ *is a first-order Lagrangian corresponding to a second-order perturbed partial differential equation (1.2). If the functional* $\int_\Omega L\, dt\, dx$ *is invariant under the one-parameter group of transformations with approximate Lie–Bäcklund symmetry generator* $X = X_0 + \epsilon X_1$*, where* $X_0 = \xi_0^1\, \partial/\partial t + \xi_0^2\, \partial/\partial x + \eta_0\, \partial/\partial u$ *and*

$X_1 = \xi_1^1\,\partial/\partial t + \xi_1^2\,\partial/\partial x + \eta_1\,\partial/\partial u$ upto gauge $B^i = B_0^i + \epsilon B_1^i$, $B^i \in \mathscr{A}$, $i = 1, 2$ then

$$X_0 L_0 + L_0 D_j\left(\xi_0^j\right) = D_j\left(B_0^j\right),$$
$$X_1 L_0 + X_0 L_1 + L_0 D_j \xi_1^j + L_1 D_j \xi_0^j = D_j B_1^j, \tag{1.8}$$

where D_i is the total differential operator with respect to x^i, i.e.,

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2.$$

In this notation, Noether's theorem reads

Theorem 2. *Corresponding to each symmetry $X = \xi^i\,\partial/\partial x^i + \eta^\alpha\,\partial/\partial u^\alpha$ that satisfies the conditions of Theorem 1, there exists an approximate conserved vector $T = (T^1, T^2)$ given by*

$$T^i = -B^i + L\xi^i + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial L}{\partial u_i^\alpha} + O(\epsilon^2) \tag{1.9}$$

such that Eq. (1.5) is satisfied.

Note. With $T^i = T_0^i + \epsilon T_1^i$, $i = 1, 2$, if we separate by powers of ϵ, order ϵ yields

$$\begin{aligned} T_1^1 &= -B_1^1 + L_0\xi_1^1 + L_1\xi_0^1 + \left(\eta_1 - u_t\xi_1^1 - u_x\xi_1^2\right)\frac{\partial L_0}{\partial u_t} \\ &\quad + \left(\eta_0 - u_t\xi_0^1 - u_x\xi_0^2\right)\frac{\partial L_1}{\partial u_t} \\ T_1^2 &= -B_1^2 + L_0\xi_1^2 + L_1\xi_0^2 + \left(\eta_1 - u_t\xi_1^1 - u_x\xi_1^2\right)\frac{\partial L_0}{\partial u_x} \\ &\quad + \left(\eta_0 - u_t\xi_0^1 - u_x\xi_0^2\right)\frac{\partial L_1}{\partial u_x} \end{aligned} \tag{1.10}$$

2. APPLICATIONS

2.1. We consider the following perturbation of the 1–1 wave equation considered by Kara *et al*. (1999),

$$u_{tt} - u_{xx} + \epsilon\left(uu_t + \frac{1}{2}tu_t^2 - \frac{1}{2}tu_x^2\right) = 0. \tag{2.1}$$

An approximate Lie point symmetry $X = X_0 + \epsilon X_1 = \partial/\partial u + \epsilon(-\frac{1}{2}tu\,\partial/\partial u)$.

X is associated with the conserved vector

$$\begin{aligned}(T^1, T^2) &= \left(T_0^1 + \epsilon T_1^1, T_0^2 + \epsilon T_1^2\right) \\ &= \left(u_t + \epsilon\left[\frac{1}{4}u^2 + \frac{1}{2}tuu_t\right], -u_x + \epsilon\left[-\frac{1}{2}tuu_x\right]\right).\end{aligned} \tag{2.2}$$

A Lagrangian corresponding to X_0 and (T_0^1, T_0^2) is $L_0 = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2$ (see Ibragimov, 1994). We use (1.10) to construct L_1, i.e., we get the system

$$\frac{\partial L_1}{\partial u_t} = \frac{1}{4}u^2 + tuu_t + B_1^1, \qquad \frac{\partial L_1}{\partial u_x} = -tuu_x + B_1^2. \tag{2.3}$$

From the first equation in (2.3), we get

$$L_1 = \frac{1}{4}u^2u_t + \frac{1}{2}tuu_t^2 + \int B_1^1\, du_t + A(t, x, u, u_x)$$

and with a choice of $B_1^1 = -\frac{1}{4}u^2$, $L_1 = \frac{1}{2}tuu_t^2 + A$. From the second equation in (2.3) and $B_1^2 = 0$, it can be shown that $A = \frac{1}{2}tuu_x^2$ so that $L_1 = \frac{1}{2}tuu_t^2 - \frac{1}{2}tuu_x^2$ so that an approximate Lagrangian for Eq. (2.1) is

$$\begin{aligned}L &= \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \epsilon\left(\frac{1}{2}tuu_t^2 - \frac{1}{2}tuu_x^2\right) \\ &= \left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2\right)(1 + \epsilon tu).\end{aligned} \tag{2.4}$$

In fact, L is equivalent to the *exact* Lagrangian $L^* = \frac{1}{2}\, e^{\epsilon tu}(u_t^2 - u_x^2)$. The Euler–Lagrange operator on L is

$$u_{tt} - u_{xx} + \epsilon\left(uu_t + \frac{1}{2}tu_t^2 - \frac{1}{2}tu_x^2\right) + \epsilon tu(u_{tt} - u_{xx}). \tag{2.5}$$

Remark. As the third term in (2.5) is of order ϵ^2, L is an approximate Lagrangian of (2.1) with approximate Noether symmetry X (as L is constructed from an approximate version of Noether's theorem), i.e., X approximately yields the corresponding variational functional invariant. Also, the vector (T_1, T_2) is an approximate Noether conserved vector.

We now study the possible existence of other approximate Noether symmetries and conserved vectors corresponding to L using (1.8) and (1.9), respectively.

2.1.1. Firstly, as $Y_0 = \partial/\partial t$ is also a Noether symmetry associated with L_0, we use the second equation in (1.8) to determine Y_1 (L_1 as obtained earlier), i.e.,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}tuu_t^2 - \frac{1}{2}tuu_x^2\right) + \left(\tau\,\frac{\partial}{\partial t} + \xi\,\frac{\partial}{\partial x} + \eta\,\frac{\partial}{\partial u} + \zeta_t\,\frac{\partial}{\partial u_t} + \zeta_x\,\frac{\partial}{\partial u_x}\right)$$

$$
\times\left(\frac{1}{2}u_t^2-\frac{1}{2}u_x^2\right)+\left(\frac{1}{2}u_t^2-\frac{1}{2}u_x^2\right)(\tau_t+u_t\tau_u+\xi_x+u_x\xi_u)
$$
$$
= D_t B_1^1 + D_x B_1^2, \tag{2.6}
$$

where $Y_1 = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}$. The procedure to determine τ and ξ (and, hence, Y_1) is standard and straightforward (e.g., see Olver, 1986); we obtain $Y_1 = -\frac{1}{4}u^2 \frac{\partial}{\partial u}$ with a choice of $B_1^1 = B_1^2 = 0$. Thus, $Y = \frac{\partial}{\partial t} + \epsilon(-\frac{1}{4}u^2\frac{\partial}{\partial u})$ is another approximate Noether symmetry corresponding to $L = L_0 + \epsilon L_1$.

From (1.9), we get $(T_0^1, T_0^2) = (-(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2), u_t u_x)$, which is well known and

$$
T_1^1 = -\frac{1}{4}u^2u_t - \frac{1}{2}tuu_t^2 - \frac{1}{2}tuu_x^2,
$$
$$
T_1^2 = \frac{1}{4}u^2u_x + tuu_tu_x,
$$

so that

$$
\left(D_t\left[T_0^1+\epsilon T_1^1\right]+D_x\left[T_0^2+\epsilon T_1^2\right]\right)\Big|_{(2.1)}
$$
$$
= -u_t\left[u_{tt}-u_{xx}+\epsilon\left(uu_t+\frac{1}{2}tu_t^2-\frac{1}{2}tu_x^2\right)\right]+\epsilon\left(\frac{1}{4}u^2+tuu_t\right)
$$
$$
\times\left[\epsilon\left(uu_t+\frac{1}{2}tu_t^2-\frac{1}{2}tu_x^2\right)\right]
$$
$$
\equiv 0.
$$

2.1.2. Similarly, as the generator $Z_0 = t\ \partial/\partial t + x\ \partial/\partial x$ is also Noether point symmetry generator of L_0, we attempt to construct an approximate Noether point symmetry $Z = Z_0 + \epsilon Z_1$ corresponding to L. This procedure yields $Z_1 = a(t,x)\ \partial/\partial t + b(t,x)\ \partial/\partial x + (-\frac{1}{4}tu^2 + c(t,x))\ \partial/\partial u$, where $a_x = b_t$, $a_t = b_x$, and $c_{tt} - c_{xx} = 0$ with $B_1^1 = -\frac{1}{12}u^3$ and $B_1^2 = 0$. As an example $Z_1 = -\frac{1}{4}tu^2$. Here,

$$
T_0^1 = -\frac{1}{2}t\left(u_t^2+u_x^2\right) - xu_tu_x,
$$
$$
T_0^2 = \frac{1}{2}x\left(u_t^2+u_x^2\right) + tu_tu_x,
$$
$$
T_1^1 = -\frac{1}{4}tu^2u_t - \frac{1}{2}t^2uu_t^2 - \frac{1}{2}t^2uu_x^2 - txuu_xu_t + \frac{1}{12}u^3,
$$
$$
T_1^2 = \frac{1}{4}tu^2u_x + \frac{1}{2}txuu_t^2 + \frac{1}{2}txuu_x^2 + t^2uu_xu_t,
$$

so that

$$
\begin{aligned}
&\left(D_t\left[T_0^1+\epsilon T_1^1\right]+D_x\left[T_0^2+\epsilon T_1^2\right]\right)\Big|_{(2.1)}\\
&\quad=-(tu_t+xu_x)\left[u_{tt}-u_{xx}+\epsilon\left(uu_t+\frac{1}{2}tu_t^2-\frac{1}{2}tu_x^2\right)\right]\\
&\qquad+\epsilon\left(\frac{1}{4}tu^2+txuu_x+t^2uu_t\right)\left[\epsilon\left(uu_t+\frac{1}{2}tu_t^2-\frac{1}{2}tu_x^2\right)\right]\\
&\quad\equiv 0.
\end{aligned}
$$

2.1.3. The symmetry $\partial/\partial x$ of L_0 yields, with L_1 as above, an approximate part $a(t,x)\,\partial/\partial t+b(t,x)\,\partial/\partial x+c(t,x)\,\partial/\partial u$, where $a_x=b_t$, $a_t=b_x$ and $c_{tt}-c_{xx}=0$.

2.2. Another perturbation of the wave equation considered by Kara *et al.* (1999) is

$$u_{tt}-u_{xx}+\epsilon u_t=0. \tag{2.7}$$

An approximate conserved vector associated with the approximate Lie point symmetry generator $X=X_0+\epsilon X_1=\partial/\partial t+\epsilon(-\frac{1}{2}u\,\partial/\partial u$ was shown to be $(T_0^1+\epsilon T_1^1,\,T_0^2+\epsilon T_1^2)$, where

$$
\begin{aligned}
T_0^1&=\frac{1}{2}u_t^2+\frac{1}{2}u_x^2,\\
T_0^2&=-u_tu_x,\\
T_1^1&=\frac{1}{2}\left(tu_t^2+tu_x^2+uu_t-uu_x\right),\\
T_1^2&=-tu_tu_x+\frac{1}{2}uu_t-\frac{1}{2}uu_x.
\end{aligned} \tag{2.8}
$$

The Lagrangian corresponding to the unperturbed equation, X_0 and (T_0^1,T_0^2) is $L_0=\frac{1}{2}(u_x^2-u_t^2)$ (this has been considered in 2.1 earlier). Proceeding as above, (1.10) yields $L_1=\frac{1}{2}t(u_x^2-u_t^2)$ so that an approximate Lagrangian is $L=\frac{1}{2}(u_x^2-u_t^2)(1+\epsilon t)$ (with $B_1^1=\frac{1}{2}uu_x$ and $B_1^2=-\frac{1}{2}uu_t$). It is again interesting to note that this an approximation of the exact Lagrangian $L^*=\frac{1}{2}e^{\epsilon t}(u_x^2-u_t^2)$. Once again, the remark regarding (approximate) Noether symmetries and invariants made in 2.1 is appropriate here.

We can, as in 2.1.1–2.1.3, determine the possible existence of other Noether symmetries and conserved vectors corresponding to L. We obtain $X=\partial/\partial x+\epsilon(a(t,x)\,\partial/\partial t+b(t,x)\,\partial/\partial x+c(t,x)\,\partial/\partial u)$, where $a_t=b_x$, $a_x=b_t$, and c satisfies the wave equation. Also, the dilation symmetry $t\,\partial/\partial t+x\,\partial/\partial x$ gives rise to the approximate part $a(t,x)\,\partial/\partial t+b(t,x)\,\partial/\partial x+(-\frac{1}{2}tu+c(t,x))\,\partial/\partial u$ with the approximate gauge term satisfying $B_1^1=-\frac{1}{4}u^2$ and $B_1^2=0$. The corresponding conserved vector has the form given by

$$
\begin{aligned}
T_0^1 &= -\frac{1}{2}t\left(u_t^2 + u_x^2\right) - xu_tu_x,\\
T_0^2 &= \frac{1}{2}x\left(u_t^2 + u_x^2\right) + tu_tu_x,\\
T_1^1 &= -\frac{1}{2}tuu_t - \frac{1}{2}t^2u_t^2 - \frac{1}{2}t^2u_x^2 - txu_tu_x + \frac{1}{4}u^2,\\
T_1^2 &= \frac{1}{2}tuu_x + \frac{1}{2}txu_t^2 + \frac{1}{2}txu_x^2 + t^2u_tu_x
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left(D_t\left[T_0^1 + \epsilon T_1^1\right] + D_x\left[T_0^2 + \epsilon T_1^2\right]\right)\big|_{(2.7)}\\
&= -(tu_t + xu_x)[u_{tt} - u_{xx} + \epsilon u_t] + \epsilon\left(\frac{1}{2}tu + txu_x + t^2u_t\right)(u_{tt} - u_{xx})\\
&\equiv 0.
\end{aligned}
$$

2.3. We now summarize equivalent results obtained for the simplest perturbation of the wave equation independent of derivative terms, viz.,

$$
u_{tt} - u_{xx} - \epsilon u = 0. \tag{2.9}
$$

Corresponding to the approximate symmetry $X = \partial/\partial t + \epsilon(a(t,x)\,\partial/\partial t + b(x,t)\,\partial/\partial x + c(t,x)\,\partial/\partial u$, where $a_t = b_x$, $a_x = b_t$, and c satisfies the wave equation, we obtain the approximate Lagrangian $L = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}\epsilon u^2$ with conserved vector components

$$
\begin{aligned}
T_0^1 &= -\frac{1}{2}\left(u_t^2 + u_x^2\right),\\
T_0^2 &= u_tu_x,\\
T_1^1 &= -\frac{1}{2}au_t^2 - \frac{1}{2}au_x^2 - bu_tu_x + \frac{1}{2}u^2,\\
T_1^2 &= \frac{1}{2}bu_t^2 + \frac{1}{2}bu_x^2 + au_tu_x.
\end{aligned}
$$

Here,

$$
\begin{aligned}
&\left(D_t\left[T_0^1 + \epsilon T_1^1\right] + D_x\left[T_0^2 + \epsilon T_1^2\right]\right)\big|_{(2.9)}\\
&= u_t[u_{tt} - u_{xx} + \epsilon u] + \epsilon(bu_x + au_t)(u_{tt} - u_{xx})\\
&\equiv 0.
\end{aligned}
$$

With $\partial/\partial t$ replaced by $\partial/\partial x$, we obtain

$$T_0^1 = \frac{1}{2}\left(u_t^2 + u_x^2\right),$$
$$T_0^2 = -u_t u_x,$$
$$T_1^1 = -\frac{1}{2}au_t^2 - \frac{1}{2}au_x^2 - bu_t u_x,$$
$$T_1^2 = \frac{1}{2}bu_t^2 + \frac{1}{2}bu_x^2 + au_t u_x + \frac{1}{2}u^2$$

with

$$\begin{aligned}&\left(D_t\left[T_0^1 + \epsilon T_1^1\right] + D_x\left[T_0^2 + \epsilon T_1^2\right]\right)\Big|_{(2.9)}\\ &= u_x[u_{tt} - u_{xx} + \epsilon u] + \epsilon(bu_x + au_t)(u_{tt} - u_{xx})\\ &\equiv 0.\end{aligned}$$

Note. The exact symmetries invloving $\partial/\partial u$ and $t\ \partial/\partial t + x\ \partial/\partial x$ do not produce perturbed conserved vectors on this Lagrangian.

2.4. An interesting example, whose first-order approximate conserved vector associated with $X = X_0 + \epsilon X_1 = \partial/\partial t + \epsilon\ \partial/\partial u$ has been determined in Kara *et al.* (1999), is

$$u_{tt} + \epsilon u_t = u_{xx}u^{\alpha}, \quad \alpha \neq -1, -2, \tag{2.10}$$

viz.,

$$\begin{aligned}T_0^1 &= -\frac{1}{2}u_t^2 - \frac{1}{(\alpha+1)(\alpha+2)}u_x^{\alpha+2},\\ T_0^2 &= \frac{1}{(\alpha+1)}u_t u_x^{\alpha+1},\\ T_1^1 &= 2\frac{\alpha+2}{3\alpha+4}xu_x u_t + \frac{\alpha+1}{3\alpha+4}uu_t - uu_t,\\ T_1^2 &= -\frac{\alpha+2}{3\alpha+4}xu_t^2 + \frac{1}{\alpha+1}uu_x^{\alpha+1} - \frac{\alpha+2}{(3\alpha+4)(\alpha+1)}uu_x^{\alpha+1} - \frac{2}{3\alpha+4}xu_x^{\alpha+2},\end{aligned} \tag{2.11}$$

provided $\alpha \neq -4/3$. We can construct a Lagrangian L_0 for the unperturbed equation using (1.9) as in Ibragimov *et al.* (1998) using X_0 and the first two equations of (2.11) to get $L_0 = \frac{1}{2}u_t^2 - \frac{1}{(\alpha+1)(\alpha+2)}u_x^{\alpha+2}$ with zero gauge B_0^1 and B_0^2. Then, we follow the procedure of example 2.1 to determine L_1. As this is tedious, we summarize the results here. With gauge terms

$$B_1^1 = -\frac{2(\alpha+2)}{3\alpha+4}u_t x u_x - \frac{2(\alpha+1)}{3\alpha+4}uu_t - \frac{1}{(\alpha+1)(\alpha+2)}tu^{\alpha+2} + u_t\left(1-\frac{1}{2}tu_t\right),$$

$$B_1^2 = \frac{\alpha+2}{3\alpha+4}xu_t^2 - \frac{2}{3\alpha+4}uu_x^{\alpha+1} + \frac{2}{3\alpha+4}xu_x^{\alpha+2} - \frac{1}{\alpha+1}u_x^{\alpha+1} + \frac{1}{\alpha+1}tu_t u_x^{\alpha+1},$$

we get $L_1 = \frac{1}{2}tu_t^2 - \frac{1}{(\alpha+1)(\alpha+2)}tu_x^{\alpha+2}$ so that an approximate Lagrangian of (2.10) is

$$L = \frac{1}{2}u_t^2 - \frac{1}{(\alpha+1)(\alpha+2)}u_x^{\alpha+2} + \epsilon\left(\frac{1}{2}tu_t^2 - \frac{1}{(\alpha+1)(\alpha+2)}tu_x^{\alpha+2}\right).$$

Note that L is a first approximation of an exact Lagrangian $L^* = e^{\epsilon t}(\frac{1}{2}u_t^2 - \frac{1}{(\alpha+1)(\alpha+2)}u_x^{\alpha+2})$.

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